2nd Midterm Review Sheet, Dr. E. Fink

Extra office hours: Monday March 13, 2:30-3:30 pm in 205G, 585 King Edward.

Tests for convergence:

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$,

(b) $\sum_{n=2}^{\infty} \frac{1}{n(ln(n))^2}$,

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$,

(d) $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i^5+1}}$,

(e) $\sum_{s=1}^{\infty} \frac{s^2-5s}{s^3+s+1}$,

(f) $\sum_{i=1}^{\infty} \frac{ln(i)}{i}$,

(g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n!}$,

(h) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$,

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$,

(j) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(k) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

(l) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

Do the following series **converge absolutely**?

(a) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+4}$

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2 n}{n^{7/8}}$

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+2}$

(e) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{\sqrt{n+2}}{\sqrt{n^2+7}}$

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$
3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$
4. $\sum_{n=1}^{\infty} \frac{n \cdot (x+1)^n}{4^n}$

Taylor and MacLaurin series

Give the MacLaurin series and their radii of convergence for the following functions. Note that often it is easier to quote the popular series or substitute into one that you know. You might also use the method of integrating or deriving a series that you already know as shown in class.

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$
5. $\arctan x$
   Recall that $\int \frac{dx}{1+x^2} = \arctan(x) + C$.
6. $\ln(1+x)$
7. $(1+x)^k$
8. $\cos(3x^2)$
9. $\ln(x^3)$
10. $e^{4/(x+3)}$
11. e^{-2x^2}
12. $x \cdot \cos(x)$
13. $\cos(x^2)$
14. $x \cdot \cos(\frac{1}{2}x^2)$

Compute the Taylor Series of the following functions at the given point $x = a$:

1. e^x at $x = 1$
2. $\ln(x+1)$ at $x = 3$
3. $\cos(x)$ at $x = \pi/2$
4. $f(x) = x^6 + 7x^4 + 3x^2 + 1$ at $x = 1$.

Do you remember these facts?

1. What is the series $\sum_{n=1}^{\infty} \frac{1}{n}$ called? Does it converge?
2. What is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ called? Does it converge?
3. What is the series $\sum_{n=0}^{\infty} q^n$ called? Does it converge? If so, what is its sum?
4. What is absolute convergence? Give an example of a series that converges but does not converge absolutely.
5. If a series converges absolutely, does it converge?
6. If a series converges, does it converge absolutely?
7. Give the formula for the Taylor Series of a function $f(x)$ centered at $x = a$.
8. Give the formula for the MacLaurin Series of a function $f(x)$.

Good luck with studying!

Solutions to the 2nd practise sheet, Dr. E. Fink

Tests for convergence:

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$: Integral test (subsitute $u = e^{-x^3}$), converges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(ln(n))^2}$: Integral test (substitute $u = (ln(x))$), converges.

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$: This sum is less than $\sum \frac{1}{k^3}$, which converges, so the series converges.

(d) $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i^5+1}}$: exponent of the a_i is $1-5/2 = 3/2$. Compare to $\sum \frac{i}{\sqrt{i^5}} = \sum \frac{1}{\sqrt{i^3}}$ which converges.

(e) $\sum_{s=1}^{\infty} \frac{s^2-5s}{s^3+s+1}$: exponent is $2-3=1$, compare to the harmonic series to show it diverges.

(f) $\sum_{i=1}^{\infty} \frac{ln(i)}{i}$: Compare to the harmonic series as for $i \geq 2$ we have $\frac{ln(i)}{i} \geq \frac{1}{i}$. Diverges.

(g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n!}$: Ratio test, converges.

(h) $\sum_{n=1}^{\infty}(-1)^{n+3}\frac{n^2}{n^3+4}$: Alternating series test shows it converges. Hint: by using the derivative of an appropriate function you can show faster that the sequence is decreasing.

(i) $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sqrt{n}}{2n+3}$: Alternating series test, converges. Use again the derivative of a function to show the sequence is decreasing for at least $n \geq 2$ (in fact $a_1 < a_2$ here, but as stated in class it is enough if it decreases for all $n > N$ for some fixed N).

(j) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$: Ratio test shows it diverges. Faster: observe that $\lim_{n\to\infty} \frac{n!}{100^n} \neq 0$, which immediately gives divergence.

(k) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$: Classic root test case, converges.

(l) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$: Alternating series test, converges conditionally. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ can be shown to be bigger than $\sum_{n=1}^{\infty} \frac{1}{5n^2}$ which diverges.

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$:

   For the radius we need to observe for which x the series converges. The form of the terms in the series suggests that the root test might be successful:

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{|(2x-1)^n|}{|5^n \cdot \sqrt{n}|}} = \lim_{n\to\infty} \frac{|2x-1|}{|5 \cdot \sqrt[n]{n}|}.$$

   Now $|2x-1|/5$ is independet of n, so we can put it in front of the limit:

$$L = \frac{|2x-1|}{5} \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}.$$

The latter limit now is 1. So we get that $L = \frac{|2x-1|}{5}$. The root test says that this converges if $L < 1$ and diverges if $L > 1$. So we check:

$$\frac{|2x-1|}{5} < 1,$$

which taking the absolute value into account is

$$\pm\frac{2x-1}{5} < 1,$$

so we have to cases:

$$\frac{2x-1}{5} < 1 \quad \text{and} \quad -\left(\frac{2x-1}{5}\right) < 1.$$

Case 1 becomes $2x-1 < 5$ or $x < 3$. Case 2 becomes $2x-1 > -5$ or $x > -2$. Now we need to check the edges of this interval, in other words does the series converge if $x = 3$ or if $x = -2$? For $x = 3$ the series is rewritten as $\sum_{n=1}^{\infty} \frac{(2\cdot 3-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We can see that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n},$$

and we know that the latter series (the harmonic series) diverges, so the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges too. Hence $x = 3$ is NOT included in the interval of convergence. For $x = -2$ we get $\sum_{n=1}^{\infty} \frac{(2\cdot(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. We can use the alternating series test to show that this series converges. So $x = -2$ is included in the interval of converges.

To sum up, the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ converges for $-2 \leq x < 3$. For the radius, we calculate the length of the interval: $3-(-2) = 5$. The the radius is the length divided by 2, so here we get $\frac{5}{2}$ as radius of convergence.

Don't forget to check the edges in any examples, it'd be a deduction of points!

2. $\sum_{n=1}^{\infty} \frac{x^n}{n\cdot 3^n}$: root test gives the inequality $|\frac{x}{3}| < 1$ from which we get the two cases $x < 3$ and $x > -3$. For $x = -3$ the series becomes the alternating harmonic series, so it converges by what we learned in class. For $x = 3$ we have the harmonic series, which diverges. Hence the interval of convergence is $-3 \leq x < 3$ and the radius is $(3-(-3))/2 = 6/2 = 3$.

3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$: Use the ratio test, for $L = \lim_{n\to\infty} \frac{\frac{|(x-2)^{n+1}|}{|(n+1)^2+1|}}{\frac{|(x-2)^n|}{|n^2+1|}} = |x-2|$. Then $|x-2| < 1$ gives $x < 3$ and $x > 1$. For the edges, for $x = 1$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ and this converges and for $x = 3$ the series converges too. So the interval of convergence is $1 \leq x \leq 3$. The radius is $(3-(1))/2 = 2/2 = 1$.

4. $\sum_{n=1}^{\infty} \frac{n\cdot(x+1)^n}{4^n}$: Root test gives the inequality $|\frac{x+1}{4}| < 1$ from which we get the two cases $x < 3$ and $x > -5$. For both edges $x = -5$ and $x = 3$ the series diverges, so the interval of convergence is $-5 < x < 3$ and the radius is $(3-(-5))/2 = 8/2 = 4$.

Taylor and MacLaurin series

Give (quoting from textbook allowed!) the MacLaurin series and their radii of convergence for:
Please take these from the table in the textbook page 762, Chapter 11.10.

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$
5. $\arctan x$
6. $\ln(1+x)$
7. $(1+x)^k$
8. $\cos(3x^2)$: Substitute $3x^2$ into the series for $\cos(x)$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n) x^{2n}}{(2n)!}$$

So

$$\cos(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 81 x^{4n}}{(2n)!}$$

9. $\ln(x^3)$
10. $e^{4/(x+3)}$
11. $e^{-2x^2} = \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n \cdot x^{2n}}{n!}$.
We use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use $-2x^2$ in place of x. Then we get $\sum_{n=0} \frac{(-2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n \cdot x^{2n}}{n!}$.
12. $x \cdot \cos(x) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \left(x \cdot \frac{(-1)^n x^{2n}}{(2n)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$.
13. $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$.
14. $x \cdot \cos(\frac{1}{2}x^2) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \left(x \cdot \frac{(-1)^n x^{4n}}{2^{2n} \cdot (2n)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{2n} \cdot (2n)!}$.

Do you remember these facts?

1. What is the series $\sum_{n=1}^{\infty} \frac{1}{n}$ called? Does it converge?

   This is the *harmonic series* and it diverges.

2. What is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ called? Does it converge?

   Alternating harmonic series, converges to $\ln(2)$.

3. What is the series $\sum_{n=0}^{\infty} q^n$ called? Does it converge? If so, what is its sum?

   Geometric series, converges if $|q| < 1$ to $\frac{1}{1-q}$.

4. What is absolute convergence? Give an example of a series that converges but does not converge absolutely.

   A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges. The series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges, but $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

5. If a series converges absolutely, does it converge?

   Yes, as this means $\sum_{n=0}^{\infty} |a_n|$ converges, and $\sum_{n=0}^{\infty} |a_n| \geq \sum_{n=0}^{\infty} a_n$, so by the comparison theorem the series converges.

6. If a series converges, does it converge absolutely?

   No. See the example of the alternating harmonic series.

7. Give the formula for the Taylor Series of a function $f(x)$ centered at $x = a$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

8. Give the formula for the MacLaurin Series of a function $f(x)$. This is the Taylor Series centered at $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Good luck with studying!